4.3 How Derivatives Effect the Shape of a Graph
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1. Overview

Increasing and Decreasing

The first derivative gives increasing/decreasing information about the original function:

\[ f'(x) \text{ positive } \rightarrow \text{slope of the tangent is positive } \rightarrow f(x) \text{ is increasing} \]
\[ f'(x) \text{ negative } \rightarrow \text{slope of the tangent is negative } \rightarrow f(x) \text{ is decreasing} \]

The only places where \( f \) can switch from increasing to decreasing are when \( f'(x) = 0 \) or \( f'(x) \text{ DNE.} \)

Note: Watch out, these numbers are not necessarily critical numbers, because critical numbers have to be in the domain of \( f(x) \)! For example \( f(x) = \frac{1}{x} \) switches from increasing to decreasing at \( x = 0 \), but \( x = 0 \) is not a critical number because it is not in the domain of \( f \).

Local Maxima and Minima

Remember from 4.1 that critical numbers are the only possibilities where local max/min may occur. (A local max/min surely must occur at a place in the domain where \( f \) switches from increasing to decreasing or decreasing to increasing.) So we find the places where local max/min occur by checking each critical number \( c \):

\[ f'(x) \text{ negative to the left of } c, f'(x) \text{ positive to the right of } c \rightarrow \text{ local min at } c \]
\[ f'(x) \text{ positive to the left of } c, f'(x) \text{ negative to the right of } c \rightarrow \text{ local max at } c \]

If the sign of the derivative is the same on both sides of \( c \), then there is neither a local min nor a local max at \( c \). This way of checking the critical numbers is called the first derivative test.

Concavity

The second derivative gives concavity information about the original function:

1. \( f''(x) \text{ positive } \rightarrow f(x) \text{ concave up} \)
2. \( f''(x) \text{ negative } \rightarrow f(x) \text{ concave down} \)

The only places where \( f \) can switch concavity are when \( f''(x) = 0 \) or \( f''(x) \text{ DNE.} \)

Inflection Points

A point \((x, y)\) on the graph of \( f(x) \) is called an inflection point if \( f \) switches concavity at \( x \). (Note that an inflection point is a point with an \( x \)-value and a \( y \)-value.)

Note: Just because a function switches concavity at \( x \), that does not mean it will have an inflection point there. For example, \( f(x) = \frac{1}{x} \), \( f(x) \) switches concavity at \( x = 0 \), but \( f(x) \) is undefined at \( x = 0 \), so there is no inflection point there.

Local Maxima and Minima Revisited

Another way to check a critical number to see if a local max/min occurs there, is by checking concavity instead of increasing/decreasing. This is called the second derivative test. You can do this as long as \( f''(c) \) exists.